Compressible Oscillating Boundary Layers

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The response of the compressible boundary layer to small fluctuations of the outer flow is investigated. The governing equations and the appropriate boundary conditions are formulated for the first time in considerable generality. It appears that the outer flow properties do not oscillate in phase with each other. Such phase differences are augmented as one proceeds across the boundary layer. Solutions are presented for small amplitudes in the form of asymptotic expansions in powers of a frequency parameter. Ordinary but coupled nonlinear differential equations are derived for self-similar flows. Results are presented for a steady part of the solution that corresponds to flat plate and stagnation flows and oscillations of the outer flow in magnitude or direction.

I. Introduction

WORK in the area of oscillating boundary layers was initiated by Moore,[1] Lighthill,[2] and Lin[3] in the early fifties. These authors have identified most of the characteristic features of such flows as, for example, the overshoot of the fluctuating part of the velocity, the phase advance or delay of properties like skin friction, heat transfer, etc. Since then a number of investigators have reconsidered the problem looking into specific flow configurations, using alternative methods of solution or investigating with more rigour the mathematical subtleties of the asymptotic expansions involved. The reader can find extensive reviews of the literature on the topic in recent review articles.[4-6] In the present paper, we are concerned with compressible boundary layers, and we will concentrate our attention only on the basic difficulties encountered in compressible flows.

The problem is posed as follows. Given the outer flow fluctuations in the velocity, find the response of the boundary layer and in particular the response of the skin friction, the heat transfer, and perhaps the location of separation. Flows through turbomachinery blades and flows over rotating helicopter blades or about fluttering wing sections are some typical examples where the present theory will find application. In fact, it is well known that, in the cases mentioned above, compressibility is one of the most significant factors. However, very little research has been carried out along this direction, and until now no one has attempted to solve the problem for a truly unsteady outer flow distribution.

Moore[1] considered first the unsteady laminar compressible boundary layer over an insulated surface. The development in this reference is for continuous time-dependent velocities of the body, and universal functions are presented from which the deviations of the velocity and temperature profiles from the quasi-steady state can be determined. Ostrach[7] and later Moore and Ostrach[8] extended the theory to include the effects of heat transfer but confined their attention to flat plate flows. The momentum equation can then be uncoupled from the energy equation, and the fact that the outer velocity has no space gradient further simplifies the problem. Illingworth[9] also considered a flat plate and studied the effect of high wall temperatures on skin friction and heat transfer due to sound waves carried by the main stream. This was actually the first paper that treated the case of fluctuating compressible outer flows in the form of an acoustic wave that accounts for fluctuations in the velocity and temperature but corresponds to a vanishing mean-pressure gradient.

Gribben[10,11] investigated the flow in the neighborhood of a stagnation point, which accounts for pressure gradient effects, albeit in a narrow sense. Gribben was mainly interested in the effects of a very hot surface and therefore simplified considerably his problem by assuming that the outer flow is incompressible and therefore assuming constant outer flow density and temperature. Vimala and Nath[12] presented most recently a quite general numerical method for solving the problem of compressible stagnation flow.

Flows with nonvanishing pressure gradients, for example flows about wedges, were studied first by Sarma.[13] Sarma employed double expansions in power of the distance along the wall and time. However, he investigated only impulsive changes of the outer flow. This introduces a considerable simplification since fluctuations in velocity and pressure are missing from the outer flow, which is thus rendered again isenthalpic. Sarma's work is further confined to adiabatic walls but accounts for suction or injection. A few years later, King[14] also considered wedge flows, but only for the case of hypersonic flow. In this particular case, the outer flow velocity may be assumed constant and only pressure fluctuations have to be taken into account. King's work is nevertheless quite interesting as it presents a perturbation method independent of the small amplitude assumption.

The present authors[15] have most recently studied the fluctuating compressible flow over a wedge or a cone but for the very special case of a wall temperature equal to the adiabatic wall temperature. To the knowledge of the authors, this is the first attempt to study a flow with nonzero pressure gradients and purely unsteady outer flow conditions. The most striking feature of such flows is the fact that the outer flow enthalpy ceases to be constant and varies proportionally to the time derivative of the pressure. However, in this work,[15] the enthalpy variations are of one order of magnitude higher than the level of the terms retained. Moreover the simplifying assumptions are quite restrictive since the solution is valid for a conducting wall with a temperature equal to the adiabatic temperature.

In the present paper, we study fluctuating flows over wedges and cones including the cases of flat plate and two-dimensional or axisymmetric stagnation flow. In this work, there is no restriction with respect to geometry (as in Refs. 1, 7-12), or with respect to the variation with time of the outer flow (as in Refs. 13 and 14) or with respect to the temperature of the wall (as in Refs. 10, 14, and 15). The formulation is based on an assumption of a small amplitude that is very realistic for a large number of practical applications. A further hypothesis that the response of the boundary layer to harmonic outer flow fluctuations is periodic, permits the

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elimination of one independent variable. The general form of the governing equations in the two-dimensional space are presented. These equations are readily available for numerical integration that requires very little storage space compared to the purely three-dimensional calculation. Such calculations have been successfully performed for the case of incompressible flow.[16] Instead we seek here solutions in terms of asymptotic expansions in powers of the frequency and the distance along the wall.

II. Governing Equations

Two-dimensional or axisymmetric laminar compressible boundary-layer flow is governed by the continuity equation, the momentum equation and the energy equation.

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^j}\frac{\partial}{\partial x}(\rho r^j u)+\frac{\partial}{\partial y}(\rho v)=0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho C_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)$$
$$=\frac{C_p}{P_r}\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

In the above equations u,v and x,y are the velocity components and the distances parallel and perpendicular to the wall, ρ, p, and T are the density, pressure, and temperature, and C_p, μ, and P_r are the specific heat for constant pressure, the viscosity, and the Prandtl number respectively. The quantity $r=r(x)$ defines the body of revolution for axisymmetric flow, and j takes the values 0 and 1 for two-dimensional and axisymmetric flow, respectively. The above system can be closed if we assume the gas to be perfect

$$p=\rho RT \tag{4}$$

The boundary conditions require, for a fixed wall with no suction

$$u=v=0 \quad \text{at } y=0 \tag{5}$$

$$T=T_W \quad \text{at } y=0 \tag{6}$$

while at the edge of the boundary layer the flow joins smoothly, within the boundary-layer approximation, with the outer flow

$$u\to U_e, \quad T\to T_e \quad \text{as } y\to\infty \tag{7}$$

where the subscript e denotes properties of the outer flow.

The outer flow distributions cannot be prescribed arbitrarily as in the case of incompressible flow. The outer flow pressure, velocity, density, and temperature distributions must be compatible and should satisfy the following equations

$$-\frac{\partial p}{\partial x}=\rho_e\left(\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}\right) \tag{8}$$

$$\frac{\partial p}{\partial t}=\rho_e C_p\left(\frac{\partial}{\partial t}+U_e\frac{\partial}{\partial x}\right)\left(T_e+\frac{U_e^2}{2C_p}\right) \tag{9}$$

and the equation of state (4). For incompressible flow, any distribution of the outer flow velocity corresponds to the evaluation at $y=0$, that is the inner limit, of some potential flow. For compressible flow, one may choose arbitrarily the distribution of one of the quantities p, ρ_e, U_e, T_e. The other three quantities, however, should be calculated from the system of Eqs. (4), (8), and (9). One further comment: the reader should notice in Eq. (9) that for unsteady flow the total enthalpy is not constant but fluctuates with the time derivative of the pressure.

A function $\Psi(x,y,t)$ is introduced according to the relationships

$$u=\frac{1}{r^j}\frac{\rho_s}{\rho}\frac{\partial}{\partial y}(r^j\Psi)=\frac{\rho_s}{\rho}\frac{\partial\Psi}{\partial y} \tag{10}$$

$$v=-\frac{\rho_s}{\rho}\left(\frac{\partial\Psi}{\partial x}+\frac{1}{r^j}\Psi\frac{\mathrm{d}r^j}{\mathrm{d}x}+\frac{\partial}{\partial t}\int_0^y\frac{\rho}{\rho_s}\mathrm{d}y\right) \tag{11}$$

where the subscript s denotes evaluation at some prescribed point of the flow. The function Ψ will be referred to in the sequel as the stream function, even though its values cannot be related directly to mass flow rates. The velocity components so defined, satisfy identically the continuity equation, and thus the system is reduced by one equation and one unknown, since Ψ can now replace u and v. A familiar transformation is then introduced[17]

$$Y=\left(\frac{p}{p_s}\right)^{1/2}\int_0^y\frac{T_s}{T}\mathrm{d}y \tag{12}$$

It was found convenient to work with a new stream function

$$\Psi(x,y,t)=(p/p_s)^{1/2}\psi(x,Y,t) \tag{13}$$

which is related to the velocity components according to the relationships

$$u=\frac{\rho_s}{\rho}\frac{\partial\Psi}{\partial y}=\frac{\partial\psi}{\partial Y} \tag{14}$$

$$v=-\frac{\rho_s}{\rho}\left(\frac{p}{p_s}\right)^{1/2}\left[\left(\frac{1}{2p}\frac{\partial p}{\partial x}+\frac{j}{r}\frac{\mathrm{d}r}{\mathrm{d}x}\right)\psi\right.$$
$$\left.+\frac{\partial\psi}{\partial x}+\frac{\partial Y}{\partial x}\frac{\partial\psi}{\partial Y}+\left(\frac{\partial Y}{\partial t}+\frac{1}{2p}\frac{\partial p}{\partial t}Y\right)\right] \tag{15}$$

Finally it is assumed that viscosity varies linearly with temperature

$$\frac{\mu}{\mu_s}=\frac{T}{T_s} \tag{16}$$

Substituting the expressions given by Eqs. (14) and (15) in Eqs. (2) and (3) and using Eqs. (8), (12), (13), and (16) we arrive at

$$\frac{\partial^2\psi}{\partial t\partial Y}+\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial x\partial Y}-\left(\frac{\partial\psi}{\partial x}+\psi\frac{j}{r}\frac{\mathrm{d}r}{\mathrm{d}x}\right)\frac{\partial^2\psi}{\partial Y^2}=\frac{T}{T_e}\left(\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}\right)-\frac{\gamma}{2a_e^2}\left(\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}\right)\psi\frac{\partial^2\psi}{\partial Y^2}+\nu_s\frac{\partial^3\psi}{\partial Y^3} \tag{17}$$

$$\frac{\partial T}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial T}{\partial x}-\left(\frac{\partial\psi}{\partial x}+\psi\frac{j}{r}\frac{\mathrm{d}r}{\mathrm{d}x}\right)\frac{\partial T}{\partial Y}=\frac{T}{T_e}\left(\frac{\partial}{\partial t}+U_e\frac{\partial}{\partial x}\right)\left(T_e+\frac{U_e^2}{2C_p}\right)-\frac{\gamma}{2a_e^2}\left(\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}\right)\psi\frac{\partial T}{\partial Y}$$
$$-\frac{T}{C_pT_e}\left(\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}\right)\frac{\partial\psi}{\partial Y}+\frac{\nu_s}{P_r}\frac{\partial^2T}{\partial Y^2}+\nu_s\left(\frac{\partial^2\psi}{\partial Y^2}\right)^2 \tag{18}$$

where γ is the ratio of the specific heats and a is the speed of sound, $a^2=\gamma RT$.

The system of equations (17, 18, and 4) can be solved for the unknown functions ψ, T, and ρ for any two-dimensional or axisymmetric unsteady compressible boundary-layer flow, if appropriate boundary conditions are imposed.

III. Small Amplitude Oscillations

The method of solution developed in the following sections can be used for a wide variety of unsteady flows. In the present paper, however, we will be concerned with fluctuating flows. In most of the earlier publications, it is assumed that the amplitude of the fluctuations is small. To remove this restriction one would have to assume instead that the flow is quasi-steady,[1,14] or employ a numerical scheme to solve the equations exactly.[12] Alternatively, one could expand in powers of the amplitude parameter ϵ and collect powers of ϵ^2 in order to capture the nonlinear effects of moderately small amplitudes. Surprisingly, this has been done only recently.[16]

We feel that for a large number of engineering applications the small amplitude assumption is perhaps the most realistic simplifying assumption. Let the outer flow velocity distribution be given by

$$U_e(x,t) = U_0(x) + \epsilon U_1(x) e^{i\omega t} \tag{19}$$

where ϵ is a small reference parameter. We will seek solutions to Eqs. (17) and (18) in the form

$$\psi(x,Y,t) = \psi_0(x,Y) + \epsilon\psi_1(x,Y) e^{i\omega t} + \ldots \tag{20}$$

$$T(x,Y,t) = T_0(x,Y) + \epsilon T_1(x,Y) e^{i\omega t} + \ldots \tag{21}$$

$$\rho(x,Y,t) = \rho_0(x,Y) + \epsilon\rho_1(x,Y) e^{i\omega t} + \ldots \tag{22}$$

Substitution of Eqs. (20-22) in Eqs. (17) and (18) and collection of terms of order ϵ^0 yields

$$\frac{\partial\psi_0}{\partial Y}\frac{\partial^2\psi_0}{\partial x\partial Y} - \left(\frac{\partial\psi_0}{\partial x} + \psi_0\frac{j}{r}\frac{\mathrm{d}r}{\mathrm{d}x}\right)\frac{\partial^2\psi_0}{\partial Y^2} = \frac{T_0 U_0 \mathrm{d}U_0/\mathrm{d}x}{T_c\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)} - \frac{\gamma U_0 \mathrm{d}U_0/\mathrm{d}x}{2a_c^2\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}\psi_0\frac{\partial^2\psi_0}{\partial Y^2} + \nu_s\frac{\partial^3\psi_0}{\partial Y^3} \tag{23}$$

$$\frac{\partial\psi_0}{\partial Y}\frac{\partial T_0}{\partial x} - \left(\frac{\partial\psi_0}{\partial x} + \psi_0\frac{j}{r}\frac{\mathrm{d}r}{\mathrm{d}x}\right)\frac{\partial T_0}{\partial Y} = -\frac{T_0 U_0 \mathrm{d}U_0/\mathrm{d}x}{C_p T_c\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}\frac{\partial\psi_0}{\partial Y} - \frac{\gamma U_0 \mathrm{d}U_0/\mathrm{d}x}{2a_c^2\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}\psi_0\frac{\partial T_0}{\partial Y}$$

$$+ \frac{T_0 U_0}{T_c\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}\frac{\partial}{\partial x}\left(T_0 + \frac{U_0^2}{2C_p}\right) + \frac{\nu_s}{P_r}\frac{\partial^2 T_0}{\partial Y^2} + \frac{\nu_s}{C_p}\left(\frac{\partial^2\psi_0}{\partial Y^2}\right)^2 \tag{24}$$

where T_c is given by

$$T_e + \frac{1}{2C_p}U_e^2 = T_c + \epsilon T_{c1}(x,t) \tag{25}$$

The quantity T_c is a constant of the energy level as Eq. (9) indicates and a_c is the speed of sound corresponding to the constant temperature T_c. The function $C_p T_{c1}(x,t)$, defined by Eq. (25), represents the fluctuating part of the enthalpy of the outer flow.

Similarly collecting powers of order ϵ^1 yields

$$\frac{i\omega\partial\psi_1}{\partial Y} + \frac{\partial\psi_1}{\partial Y}\frac{\partial^2\psi_0}{\partial x\partial Y} + \frac{\partial\psi_0}{\partial Y}\frac{\partial^2\psi_1}{\partial x\partial Y} - \left(\frac{\partial\psi_0}{\partial x} + \psi_0\frac{j}{r}\frac{\mathrm{d}r}{\mathrm{d}x}\right)\frac{\partial^2\psi_1}{\partial Y^2} - \left(\frac{\partial\psi_1}{\partial x} + \psi_1\frac{j}{r}\frac{\mathrm{d}r}{\mathrm{d}x}\right)\frac{\partial^2\psi_0}{\partial Y^2}$$

$$= \frac{T_0}{T_c\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}\left[\frac{\mathrm{d}}{\mathrm{d}x}(U_0U_1) + i\omega U_1\right] + T_1\left[1 - \frac{\gamma-1}{2a_c^2}U_0^2\right] - \frac{T_0\left(T_{c1}/T_c - \frac{\gamma-1}{a_c^2}U_0U_1\right)}{T_c\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)^2}U_0\frac{\mathrm{d}U_0}{\mathrm{d}x}$$

$$- \frac{\gamma}{2a_c^2\left(1 - \frac{\gamma-1}{2a_c^2}U_0^2\right)}\left\{\left(\psi_0\frac{\partial^2\psi_1}{\partial Y^2} + \psi_1\frac{\partial^2\psi_0}{\partial Y^2}\right)U_0\frac{\mathrm{d}U_0}{\mathrm{d}x} \quad \psi_0\frac{\partial^2\psi_0}{\partial Y^2}\left[\frac{\mathrm{d}}{\mathrm{d}x}(U_0U_1) + i\omega U_1\right]\right\}$$

$$- \frac{T_{c1}/T_c - \frac{\gamma-1}{a_c^2}U_0U_1}{1 - \frac{\gamma-1}{2a_c^2}U_0^2}U_0\frac{\mathrm{d}U_0}{\mathrm{d}x}\psi_0\frac{\partial^2\psi_0}{\partial Y^2} + \nu_s\frac{\partial^3\psi_1}{\partial Y^3} \tag{26}$$

$$i\omega T_1+\frac{\partial\psi_0}{\partial Y}\frac{\partial T_1}{\partial x}+\frac{\partial\psi_1}{\partial Y}\frac{\partial T_0}{\partial x}-\left(\frac{\partial\psi_0}{\partial x}+\psi_0\frac{j}{r}\frac{dr}{dx}\right)\frac{\partial T_1}{\partial Y}-\left(\frac{\partial\psi_1}{\partial x}+\psi_1\frac{j}{r}\frac{dr}{dx}\right)\frac{\partial T_0}{\partial Y}=\frac{T_0(i\omega T_{c1}+U_0 dU_0/dx)}{T_c(1-\frac{\gamma-1}{2a_c^2}U_0^2)}$$

$$-\frac{\gamma}{2a_c^2(1-\frac{\gamma-1}{2a_c^2}U_0^2)}\left\{\left[\psi_0\frac{\partial T_0}{\partial Y}\left(\frac{d}{dx}(U_0U_1)-i\omega U_1\right)+U_0\frac{dU_0}{dx}\left(\psi_0\frac{\partial T_1}{\partial Y}+\psi_1\frac{\partial T_0}{\partial Y}\right)\right]\right.$$

$$\left.-\frac{U_0\frac{dU_0}{dx}\psi_0\frac{\partial T_0}{\partial Y}\left(\frac{T_{c1}}{T_c}-\frac{\gamma-1}{a_1^2}\right)^2U_1U_0\cos\omega t}{1-\frac{\gamma-1}{2a_c^2}U_0^2}\right\}-\frac{T_0}{C_pT_c(1-\frac{\gamma-1}{2a_c^2}U_0^2)}\left[\frac{\partial\psi_1}{\partial Y}U_0\frac{dU_0}{dx}+\frac{\partial\psi_0}{\partial Y}\left(\frac{d}{dx}(U_0U_1)-i\omega U_1\right)\right] \quad (27)$$

where T_c is a constant of integration (see Eq. (25).

IV. Outer Flow and Boundary Conditions

If the outer flow velocity distribution is prescribed, then the outer pressure density and temperature distributions can be calculated from Eqs. (4), (8), and (9). For a small amplitude fluctuation given by Eq. (19), we expect that the other properties of the outer flow will behave similarly.

$$T_e(x,t)=T_{e0}(x)+\epsilon T_{e1}(x)e^{i\omega t}+\ldots \quad (28)$$

$$\rho_e(x,t)=\rho_{e0}(x)+\epsilon\rho_{e1}(x)e^{i\omega t}+\ldots \quad (29)$$

$$\rho(v,\epsilon)=\rho_0(v)+\epsilon p_1(x)e^{i\omega t}+\ldots \quad (30)$$

In terms of these quantities, the boundary conditions at the edge of the boundary layer become

$$\frac{\partial\psi_0}{\partial Y}\to U_0,\quad \frac{\partial\psi_1}{\partial Y}\to U_1 \quad \text{as } Y\to\infty \quad (31)$$

$$T_0\to T_{e0},\quad T_1\to T_{e1} \quad \text{as } Y\to\infty \quad (32)$$

The no-slip and no-penetration conditions at the wall require

$$\frac{\partial\psi_0}{\partial Y}=0,\quad \frac{\partial\psi_1}{\partial Y}=u_w(t) \quad \text{at } Y=0 \quad (33)$$

$$\psi_0=\psi_1=0 \quad \text{at } Y=0 \quad (34)$$

and finally at the wall we demand that

$$T=T_w \quad \text{at } Y=0 \quad (35)$$

where u_w and T_w are the wall velocity and temperature, respectively.

Substitution of Eqs. (28-30) and Eq. (19) into Eqs. (4), (8), and (9) and collection of terms of order ϵ^0 yields

$$-\frac{\partial p_0}{\partial x}=\rho_{e0}U_0\frac{dU_0}{dx} \quad (36)$$

$$o=\rho_{e0}U_0\frac{d}{dx}\left(T_{e0}+\frac{U_0^2}{2C_p}\right) \quad (37)$$

$$p_0=\rho_{e0}RT_{e0} \quad (38)$$

The solution to the above system is

$$T_{e0}=T_c-U_0^2/2C_p \quad (39)$$

$$p_0/p_s=(T_c-U_0^2/2C_p)\gamma/\gamma-1 \quad (40)$$

$$\rho_{e0}=\frac{p_s}{R}(T_c-U_0^2/2C_p)1/\gamma-1 \quad (41)$$

Collecting now terms of order ϵ, we get after rearranging

$$\frac{dp_1}{dx}=-p_1\frac{U_0}{T_{e0}R}\frac{dU_0}{dx}+T_{e1}\frac{\rho_{e0}U_0}{T_0}\frac{dU_0}{dx}-\rho_0[i\omega U_1+\frac{d}{dx}(U_0U_1)] \quad (42)$$

$$\frac{dT_{e1}}{dx}=p_1\frac{i\omega}{C_p\rho_{e0}U_0}-i\omega T_{e1}-\frac{i\omega U_1}{C_p}-\frac{1}{C_p}\frac{d}{dx}(U_0U_1) \quad (43)$$

$$p_1=R(\rho_{e0}T_{e1}+\rho_{e1}T_{e1}) \quad (44)$$

We were unable to find a closed form solution to the above system of equations. However, a numerical integration can accompany the matching of a numerical solution of the boundary-layer equations and thus provide at each step the boundary condition at the edge of the boundary layer. In this way, the outer inviscid flow equations can be solved simultaneously with the boundary-layer equations. In the present paper, we will present analytical solutions in the form of coordinate expansions in powers of $\omega x/U_\infty$.

Let

$$p_1=\sum_{n=0}^{\infty}(i\omega)^np_{1n},\quad T_{e1}=\sum_{n=0}^{\infty}(i\omega)^nT_{1n}$$

$$\rho_{e1}=\sum_{n=0}^{\infty}(i\omega)^n\rho_{1n} \quad (45)$$

Once again the familiar perturbation process, assuming that ω is small, gives an infinite sequence of sets of differential equation. The first two sets are

$$\frac{dp_{10}}{dx}=-p_{10}\frac{U_{e0}}{T_{e0}R}\frac{dU_0}{dx}+T_{10}\frac{\rho_{e0}U_0}{T_{e0}}\frac{dU_0}{dx}-\rho_{e0}\frac{d}{dx}(U_0U_1) \quad (46)$$

$$\frac{dT_{10}}{dx}=-\frac{1}{C_p}\frac{d}{dx}(U_0U_1) \quad (47)$$

$$p_{10}=R(\rho_{e0}T_{10}+\rho_{10}T_{e0}) \quad (48)$$

and

$$\frac{dp_{11}}{dx}=p_{11}\frac{U_{e0}}{T_{e0}R}\frac{dU_0}{dx}+T_{11}\frac{\rho_{e0}U_{e0}}{T_{e0}}\frac{dU_0}{dx}-\rho_{e0}U_1 \quad (49)$$

$$\frac{dT_{11}}{dx}=\frac{1}{C_p\rho_{e0}U_0}p_{10}-\frac{T_{10}}{U_0}-\frac{U_1}{C_p} \quad (50)$$

$$p_{11}=R(\rho_{e0}T_{11}+\rho_{11}T_{e0}) \tag{51}$$

while for $n \geq 2$ we get

$$\frac{\mathrm{d}p_{1n}}{\mathrm{d}x}=p_{1n}\frac{U_0}{T_{e0}R}\frac{\mathrm{d}U_0}{\mathrm{d}x}+T_{1n}\frac{\rho_{e0}U_0}{T_{e0}}\frac{\mathrm{d}U_0}{\mathrm{d}x} \tag{52}$$

$$\frac{\mathrm{d}T_{1n}}{\mathrm{d}x}=\frac{1}{C_p\rho_{0e}U_0}p_{1,n-1}-\frac{1}{U_0}T_{1,n-1} \tag{53}$$

$$p_{1n}=R(\rho_{e0}T_{1n}+\rho_{1n}T_{e0}) \tag{54}$$

The system of Eqs. (46) through (48) has a closed form solution

$$p_{10}=-\rho_{e0}U_0U_1 \tag{55}$$

$$T_{10}=-U_0U_1/C_p \tag{56}$$

$$\rho_{10}=-\frac{\rho_{e0}U_0U_1}{(\gamma-1)T_{e0}} \tag{57}$$

The solution of the other systems of equations is quite straightforward but involves expressions like $\int U_1 \mathrm{d}x$ and may appear a little involved.

In the present paper, consider solutions for outer flow velocity distributions given by

$$U_0=c_0x^m \qquad U_1=c_1x^{\ell} \tag{58}$$

In terms of the above expressions the solution given by Eqs. (39) and (41) becomes

$$T_{e0}=T_c-\frac{c_0^2x^{2m}}{2C_p} \tag{59}$$

$$p_0/p_s=\left(T_c-\frac{c_0^2x^{2m}}{2C_p}\right)^{\gamma/\gamma-1} \tag{60}$$

$$\rho_{e0}=\frac{p_s}{R}\left(T_c-\frac{c_0^2x^{2m}}{2C_p}\right)^{1/\gamma-1} \tag{61}$$

and the solution of order ϵ given by Eqs. (55) and (57) becomes

$$p_{10}=-\frac{p_s}{R}\left(T_c-\frac{c_0^2x^{2m}}{2C_p}\right)^{1/\gamma-1}c_0c_1x^{m+\ell} \tag{62}$$

$$T_{10}=-\frac{c_0c_1}{C_p}x^{m+\ell} \tag{63}$$

$$\rho_{10}=-\frac{p_s}{R(\gamma-1)}\left(T_c-\frac{c_0^2x^{2m}}{2C_p}\right)^{1/\gamma-1} c_0c_1x^{m+\ell}\left(T_c-\frac{c_0^2x^{2m}}{2C_p}\right)^{-1} \tag{64}$$

Finally, the first correction that accounts for terms of order ω, that is the solution to Eqs. (49) and (51), for the special case of power variations of U_0 and U_1, can be written as follows

$$p_{11}=-\frac{c_1x^{\ell+1}p_0}{(\ell+1)RT_0} \tag{65}$$

$$T_{11}=-\frac{c_1x^{\ell+1}}{C_p(\ell+1)} \tag{66}$$

$$\rho_{11}=-\frac{\rho_{e0}x^{\ell+1}}{RT_{e0}(\ell+1)\gamma(\gamma-1)} \tag{67}$$

Notice that the fluctuating part of the pressure, temperature, and density are not in phase with the velocity. Retaining terms of order ϵ and ω only, we can write the fluctuating part of the pressure and temperature as follows

$$\frac{p_1}{p_0}=-\frac{U_0U_1}{RT_0}\left[1+\frac{i\omega x}{U_0(\ell+1)}\right] \tag{68}$$

$$T_{e1}=-\frac{U_0U_1}{C_p}\left[1+\frac{i\omega x}{U_0(\ell+1)}\right] \tag{69}$$

Therefore the pressure and temperature lead the outer flow velocity by the same phase angle. This should not be confused with phase differences that appear within the boundary layer. The reader should recall for example that even for incompressible flow with pressure and otuer flow temperature fluctuating in phase with the outer flow velocity, the skin friction and wall heat transfer usually lead and lag respectively in their periodic variation, the outer flow.

V. Method of Solution

A new independent variable, familiar from self-similar flows is introduced[17]

$$\eta=\left[\frac{(m+1+2j)U_0}{2\nu_s x}\right]^{1/2}Y \tag{70}$$

Solutions to Eqs. (23) and (24) will be sought in the form of series expansions

$$\psi_0(x,Y,t)=c_0\left[\frac{2\nu_s x}{(m+1+2j)U_0}\right]^{1/2}\sum_{n=0}^{\infty}\frac{c_0^{2n}}{a_c^{2n}}x^{(1+2n)}f_{1+2n}(\eta) \tag{71}$$

$$T_0(x,Y,t)=T_c\left(1-\frac{\gamma-1}{2}\frac{U_0^2}{a_c^2}\right)\sum_{n=0}^{\infty}\frac{c_0^{2n}}{a_c^{2n}}x^{2mn}g_{1+2n}(\eta) \tag{72}$$

The steady-state problem for compressible flow over wedges or cones has been studied extensively in the literature.[18] It should be noticed that the above formulation holds for arbitrary values of the Prandtl number, the ratio of specific heats, and the pressure gradient. For $m=0$, that is for a flat plate the series expansion is essentially an expansion in powers of the Mach number. For flows with pressure gradient, however, the above expressions becomes coordinate expansions and hold for any value of the Mach number but not very far from the leading edge of the wedge or the cone. These expressions are substituted in the differential Eqs. (23) and (24) and equal powers of x, that is $2mn+m-1$ powers for $n=0,1,\ldots$, are collected. The following ordinary but coupled differential equations are then derived

$$f_1'''=-f_1f_1''-\left(\frac{2m}{m+1+2j}\right)\left(g_1-f_1'^2\right) \tag{73}$$

$$f_{1+2n}'''=2\sum_{q=0}^{n}\frac{2mq+m}{m+1+2j}f_{1+2q}'f_{1+2n-2q}' - \sum_{q=0}^{n}\frac{4mq+m+1+2j}{m+1+2j}f_{1+2q}f_{1+2n-2q}''-\frac{2m}{m+1+2j}g_{1+2n} + \gamma\frac{m}{m+1+2j}\sum_{s=0}^{n-1}\left(\frac{\gamma-1}{2}\right)^{n-s-1}\left(\sum_{q=0}^{s}f_{1+2q}f_{1+2s-2q}''\right) \tag{74}$$

for $n \geq 1$

$$g_1''=-P_rf_1g_1' \tag{75}$$

$$\frac{1}{P_r} g''_{1+2n} - \frac{\gamma-1}{2P_r} g''_{1+2(n-1)} = \frac{4m}{m+1+2j} \sum_{q=0}^{n} q f'_{1+2n-2q} g_{1+2q}$$

$$- \frac{2m(\gamma-1)}{m+1+2j} \sum_{q=0}^{n-1} q f'_{1+2(n-1-q)} g_{1+2q}$$

$$- \frac{1}{m+1+2j} \sum_{q=0}^{n} [4m(n-q)+m+1+2j] f_{1+2(n-q)} g'_{1+2q}$$

$$+ \frac{1}{m+1+2j} \sum_{q=0}^{n-1} \left\{ \gamma m + \frac{\gamma-1}{2} [4m(n-q-1)+m+1+2j] \right\}$$

$$\times f_{1+2(n-q-1)} g'_{1+2q} - (\gamma-1) \sum_{q=0}^{n-1} f''_{1+2q} f''_{1+2(n-1-q)} \qquad (76)$$

for $n \geq 1$

The boundary conditions in terms of the functions f and g become

$$f_{1+2n}(0) = f'_{1+2n}(0) = 0 \qquad \text{for } n \geq 0 \qquad (77)$$

$$g_1(0) = T_{w0}, \qquad g_{1+2n}(0) = 0 \qquad \text{for } n \geq 1 \qquad (78)$$

$$f'_1(\infty) \to 1, \qquad g_1(\infty) \to 1 \qquad (79)$$

$$f'_{1+2n}(\infty) \to 0, \qquad g_{1+2n}(\infty) \to 0 \qquad \text{for } n \geq 1 \qquad (80)$$

The constant quantity of T_{w0} is a dimensionless temperature related to the wall temperature via Eq. (72). $T_{w0} = 0$, 1, and >1 then corresponds to a cold wall, zero temperature difference, and a hot wall, respectively.

Similarly, solutions to Eqs. (26) and (27) will be sought in the form

$$\psi_1(x,Y,t) = \left[\frac{2\nu_s x}{(m+1+2j)U_0}\right]^{1/2} c_1 \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \frac{c_0^{2n}}{a_c^{2n}}$$

$$x^{2mn+\ell+(1-m)q} \left(\frac{i\omega}{c_0}\right)^q F_q^{(2n)}(\eta) \qquad (81)$$

$$T_1(x,Y,t) = \frac{c_0 c_1}{C_p} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \frac{c_0^{2n}}{a_c^{2n}} x^{2mn+\ell+m+(1-m)q}$$

$$\times \left(\frac{i\omega}{c_0}\right)^q G_q^{(2n)}(\eta) \qquad (82)$$

The above expressions are substituted in Eqs. (26) and (27), and equal powers of x are collected. For $n=0$ the general form of the differential equations for $F_q^{(0)}$ and $G_q^{(0)}$ are

$$F_q'''^{(0)} = -f_1 F_q''^{(0)} + \frac{m+\ell+(1-m)q}{m+1+2j} 2f'_1 F_q'^{(0)}$$

$$- \frac{2\ell+2(1-m)q+(1-m)+2j}{m+1+2j} f''_1 F_q^{(0)} + \frac{2}{m+1+2j} F'^{(0)}_{q-1}$$

$$- \frac{2}{m+1+2j} [(m+\ell)\delta_{0q} + \delta_{1q}] g_1 \qquad (83)$$

$$\frac{1}{P_r} G_q''^{(0)} = \frac{2(\ell+m+(1-m)q)}{m+1+2j} G_q^{(0)} f'_1$$

$$+ \frac{4m}{(\gamma-1)(m+1+2j)} g_3 F_q'^{(0)} - \frac{2m}{(m+1+2j)} F_q'^{(0)} g_1$$

$$-f_1 G_q'^{(0)} - \frac{1}{(\gamma-1)(m+1+2j)} \Big\{ (2\ell+2c_1-m)q$$

$$+ (1-m)+2j) F_q^{(0)} g'_3 + (4m+2\ell+2(1-m)q+(1-m)$$

$$+2j) F_q^{(2)} g'_1 \Big\} + \frac{1}{2} \left(\frac{2\ell+2(1-m)q+(1-m)+2j}{m+1+2j}\right) F_q^{(0)} g'_1$$

$$+ \frac{2g_1}{(m+1+2j)} \left(\frac{1}{\ell+1}\delta_{2q} + \delta_{1q}\right) + \frac{\gamma f_1 f'_1 (\ell+m)\delta_{0q} + \delta_{1q})}{(\gamma-1)(m+1+2j)}$$

$$+ \frac{\gamma m}{(m+1+2j)(\gamma-1)} F_q^{(0)} g'_1 + \frac{2m}{(m+1+2j)} g_1 F_q'^{(0)}$$

$$+ \frac{2}{(m+1+2j)} g_1 f'_1 ((m+\ell)\delta_{0q} + \delta_{1q}) - f''_1 F_q''^{(0)}$$

$$+ \frac{2}{(m+1+2j)} G^{(0)}_{q-1} \qquad (84)$$

where δ_{ij} is Kronecker's delta. The equations for $F_q^{(2)}$ and $G_q^{(2)}$ are given in the appendix.

The boundary conditions in terms of the functions F and G read

$$F_q^{(2n)}(0) = F_q'^{(2n)}(0) = 0 \qquad (85)$$

$$G_0^{(0)}(0) = 1, \quad G_q^{(2n)}(0) = 0 \text{ for } q \geq 1, \quad n \geq 0 \qquad (86)$$

$$F_0'^{(0)}(\infty) \to 1, \quad F_q'^{(2n)}(\infty) = 0 \text{ for } q,n \geq 1 \qquad (87)$$

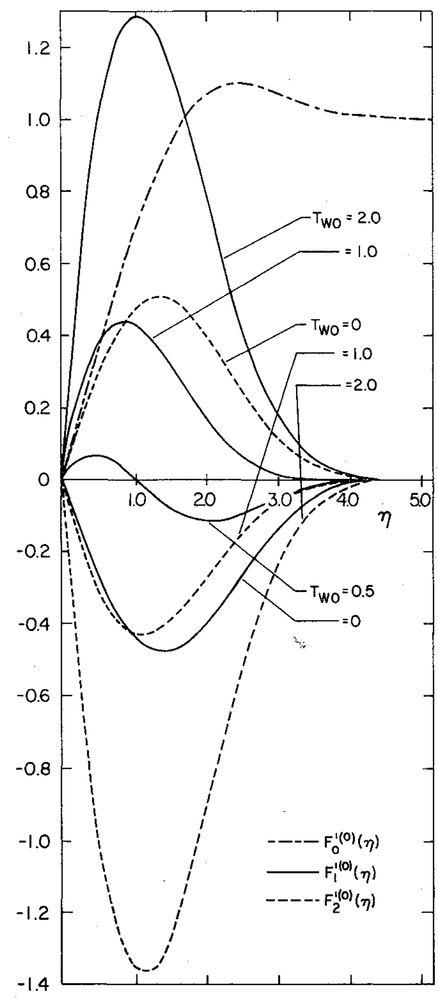

Fig. 1 The profiles of the functions $F_0'^{(0)}$, $F_1'^{(0)}$, and $F_2'^{(0)}$ for $m=0$, $\ell=0$, and different values of the wall temperature.

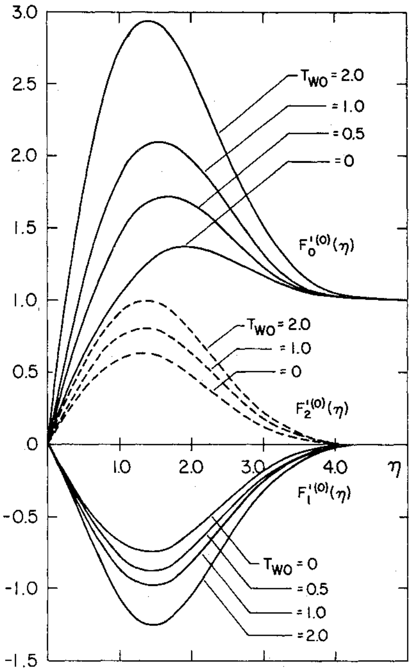

Fig. 2 The profiles of the functions $F_0'^{(0)}$, $F_l'^{(0)}$, and $F_2'^{(0)}$ for $m=0$, $\ell=1$ and different values of the wall temperature.

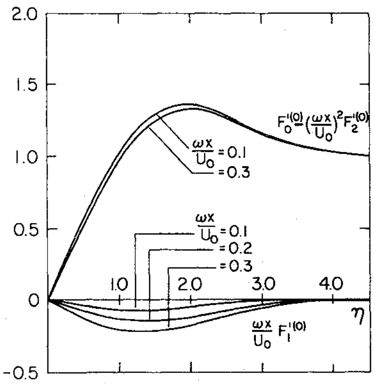

Fig. 3 The in-phase and out-of-phase parts of the fluctuating velocity for $m=0$, $\ell=1$, $T_{w0}=0$ and different values of the frequency parameter.

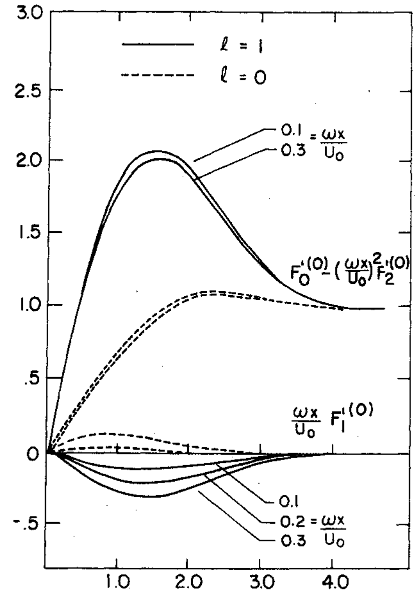

Fig. 4 The in-phase and out-of-phase parts of the fluctuating velocity profile for $m=0$, $T_{w0}=1.0$ and different values of the frequency parameter.

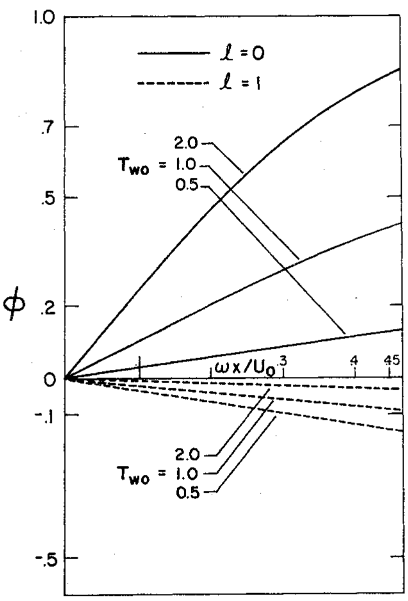

Fig. 5 The skin-friction phase angle as a function of the frequency parameter for $m=0$.

$$G_0^{(0)}(\infty)\rightarrow -1,\quad G_l^{(0)}(\infty)\rightarrow -\frac{l}{\ell+1},\quad G_q^{2n}(\infty)\rightarrow 0 \tag{88}$$

$$\text{for}\quad q\geq 2,\quad n\geq 0$$

The ordinary differential equations were solved numerically by the shooting technique. Each system of coupled equations as for example the system for f_l, g_l or f_3, g_3, etc. is replaced by systems of first-order differential equations. Two more boundary conditions at the wall are assumed and the problem is solved as an initial value problem. A straightforward fourth-order Runge-Kutta integration scheme is employed and the values of the functions at the edge of the boundary layer are checked against the outer flow boundary conditions. If these conditions are not met, a guided guess is used to readjust the assumed values at the wall and the process is repeated until convergence is achieved.

VI. Results and Discussion

In this paper, we present numerical results only for $P_r=0.72$ and for the cases of $m=0$ and $m=1$, that is only for a steady part of the flow that corresponds to compressible flow over a flat plate or stagnation flow. The method is by no means confined to these values of m and our subroutine can produce results for any intermediate values that correspond to wedges and cones of different included angles. In the case of $m=0$ the zeroth order problem is simplified considerably since the momentum equation is uncoupled from the energy equation. Nevertheless, some interesting features of the unsteady part of the flow are present, especially in the case of outer flow velocity fluctuations that vary with a power of x, $U_l=c_l x^\ell$. Physically, this corresponds to a flat plate that oscillates and bends as if it were pivoted at its leading edge.

The differential equations that govern the functions f_l and g_l were solved numerically as described in the previous section. The results were checked against previous theoretical data but are omitted here due to lack of space. The same subroutine was used to solve numerically the differential equations that govern the functions $F_q^{(0)}$ for $q=0,1,2$ and $G_0^{(0)}$. In Fig. 1, we have plotted the functions $F_0'^{(0)}$, $F_l'^{(0)}$, and $F_2'^{(0)}$ for $\ell=0$, that is $U_l=c_l$ and different wall temperatures.

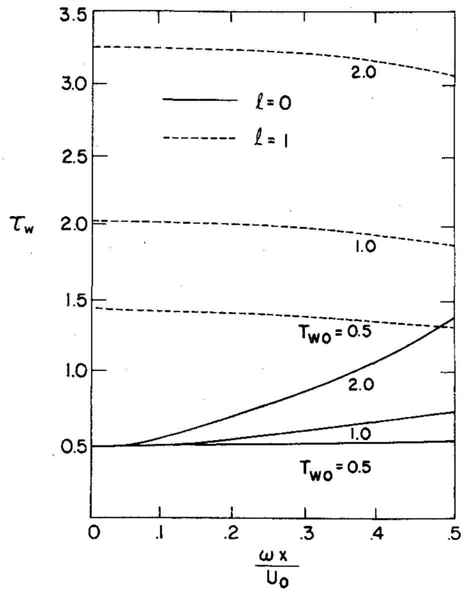

Fig. 6 The dimensionless skin friction as a function of the frequency parameter for $m = 0$.

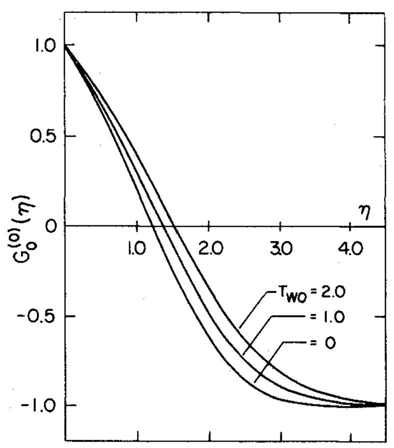

Fig. 7 The profile of the temperature function $G_0^{(0)}$ for $\ell=0$ and different values of the wall temperature.

This corresponds to a change of the free stream in magnitude and not in direction. If $\ell=0$, the function $F_0^{(0)}$ is independent of the wall temperature, T_{w0}, and it clearly indicates an overshoot, a characteristic feature of the fluctuating component of the velocity. It is interesting to note that the other functions are strongly affected by the wall temperature. In fact, comparing results for hot walls and hot streams, it appears that the trend can be completely reversed. This is not true for $\ell=1$, that is for $U_l = c_l x$, as Fig. 2 shows. Apparently, the velocity fluctuations override the effect of the hot stream, and the trend of the curves is more uniform. In this case, wall temperature appears to affect violently the amplitude of the velocity fluctuations at the level of $F_0^{(0)}$ as well. The reader should notice that the overshoot of $F_0'^{(0)}$ is now much stronger than in the case $U_l = c_l$.

In Figs. 3 and 4, we have collected some of the calculated terms of the velocity expansions for a stream fluctuating in magnitude, $\ell=0$, and a stream fluctuating in direction $\ell=1$, respectively. In the traditional terminology, we call the real and imaginary parts of the fluctuating velocity, in-phase $u_{\rm in}(x,\eta) = F_0'^{(0)}(\eta) - (\omega x/U_0)^2 F_2'^{(0)}(\eta)$ and out-of-phase velocity components $u_{\rm out}(x,\eta) = \omega x/U_0) F_1'^{(0)}(\eta)$. The reader may observe a stronger overshoot for streams fluctuating in a direction over a hot wall. The phase angle profile can be readily calculated from Figs. 3 and 4.

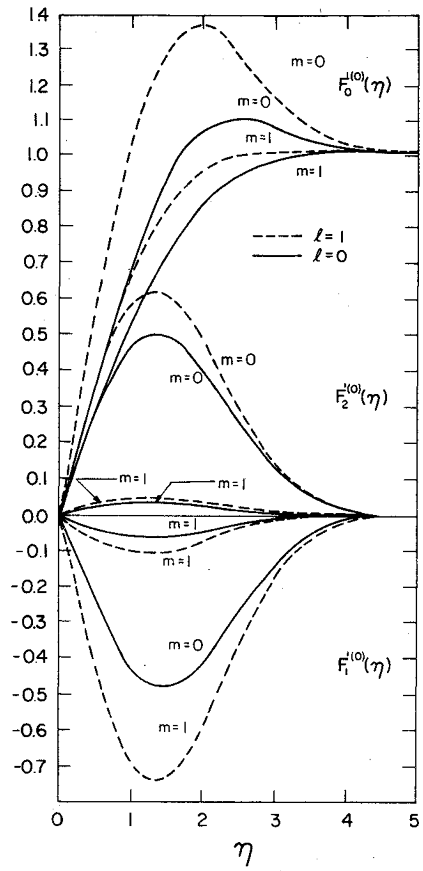

Fig. 8 The profiles of the functions $F_0'^{(0)}$, $F_1'^{(0)}$, and $F_2'^{(0)}$ for flat plate and stagnation flow ($m = 0$ and 1) and $T_{w0} = 0$.

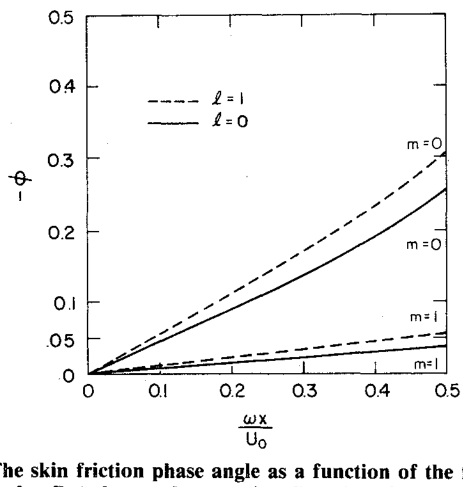

Fig. 9 The skin friction phase angle as a function of the frequency parameter for flat plate and stagnation flow ($m=0$ and 1) and $T_{w0} = 1.0$.

In Fig. 5, we have plotted the skin friction phase angle as a function of the frequency parameter. It should be noticed that for a stream fluctuating in magnitude the skin friction is lagging the outer stream flow. Figure 6 is a plot of the variation of the dimensionless skin friction vs the frequency parameter $\omega x/U_0$. It appears that no considerable changes are involved, except perhaps for a fluctuating plate and a hot wall. Finally, in Fig. 7, we have plotted the unsteady part of order ω° of the temperature function $G_0^{(0)}$ for three different values of the wall temperature T_{w0}.

Calculations were also performed for $m=1$, that is, for stagnation mean flow with imposed fluctuations. Figure 8 shows the functions $F_0'^{(0)}$, $F_1'^{(0)}$, $F_2'^{(0)}$. The overshoot of the in-phase velocity profile is shown to be greatly diminished as compared to the flat-plate case, $m=0$. The function $F_1'^{(0)}$, which strongly affects the out-of-phase velocity component and therefore the phase angles, appears to be considerably smaller. From these results and results for intermediate values of m that are omitted due to lack of space, we concluded that the phase function decreases as the pressure gradient increases. This is clearly observed in Fig. 9, in which the phase advance is plotted as a function of the frequency function. It appears that for more adverse pressure gradients that is for m increasing from 0 and approaching 1, the overshoot of the velocity profile and the phase angles are suppressed.

VII. Conclusions

A great number of papers has appeared in the literature on unsteady and usually oscillating boundary layers. In all of these publications, the outer flow disturbance is assumed in the form of a correction to the mean velocity. In the present paper, it is indicated that for compressible flow, with time-dependent pressure gradients, the inviscid flow properties must be compatible with each other. This generates, in the outer flow, phase differences among the velocity, pressure, and temperature. These phase differences are augmented as one traverses the boundary layer.

The differential equations derived in Sec. III represent the mean flow and the unsteady perturbation for a small amplitude of oscillation. These equations can be easily solved numerically in the two-dimensional space for any body configuration and any values of the frequency. Elimination of time in this case drastically reduces the required computer space and time. This method has been followed by one of the authors, quite successfully, for incompressible flow. Instead, in this paper we further reduce the order of the differential equations by virtue of expansions whose first term is the classical self-similar flow about a wedge. Moreover, the unsteady disturbance is calculated only for small values of the frequency, although large values of frequency can be considered also by a similar method.

The double expansions of Sec. V contain the correction of compressibility on the unsteady part of the motion. It is interesting to note that for $n=0$, the expansions represent the incompressible disturbance on a compressible mean field. Compressibility appears to affect significantly the fluctuating components of the velocity and temperature. It is indicated that overshoots and wall slopes of velocity and temperature can be increased significantly as compared to incompressible flow. This implies larger fluctuating values of the skin friction and heat transfer. Finally it is again verified that larger favorable pressure gradients suppress the phenomena of unsteadiness.

Appendix

The differential equations for the functions $F_q^{(2)}$ and $G_q^{(2)}$ of the expansions (81) and (82), respectively, are

$$
\begin{aligned}
F_q'''^{(2)} = & -f_1 F_q''^{(2)} - \left(\frac{5m+1+2j}{m+1+2j}\right) f_3\, F_q''^{(0)} + 2\left(\frac{2m+\ell+(1-m)q+1}{m+1+2j}\right) f_1'\, F_q'^{(2)} + 2\left(\frac{3m+\ell+(1-m)q}{m+1+2j}\right) f_3'\, F_q'^{(0)} \\
& - \left(\frac{2\ell+2(1-m)q+(1-m)+2j}{m+1+2j}\right) f_3''\, F_q^{(2)} - \left(\frac{4m+2\ell+2(1-m)q+(1-m)+2j}{m+1+2j}\right) f_1''\, F_q^{(2)}(\eta) \\
& - 2\left(\frac{(m+\ell)\delta_{0q}+\delta_{1q}}{m+1+2j}\right) g_3 - \frac{2m(\gamma-1)}{m+1+2j} G_q^{(0)}(\eta) - \frac{2m(\gamma-1)}{m+1+2j}\left(\delta_{0q} + \frac{1}{\ell+1}\delta_{1q}\right) g_1 + \frac{\gamma m}{m+1+2j} f_1\, F_q''^{(0)} \\
& + \frac{\gamma m}{m+1+2j} f_1'' F_q^{(0)} + \frac{\gamma}{m+1+2j}\left((m+\ell)\delta_{0q}+\delta_{1q}\right) f_1\, f_1''
\end{aligned} \tag{A1}
$$

$$
\begin{aligned}
\left(\frac{m+1+2j}{2}\right)\frac{1}{P_r} G_q''^{(2)} = & \,(5m+\ell+(1-m)q)\, f_1' G_q^{(2)} - \left(\frac{m+1+2j}{2}\right) f_1 G_q'^{(2)} + (\ell+m+(1-m)q)\, f_3' G_q^{(0)} \\
& - \left(\frac{5m+1+2j}{2}\right) f_3 G_q'^{(0)} + \frac{1}{(\gamma-1)}\left\{4mF_q'^{(0)} g_5 + 2mg_3 F_q'^{(2)}\right\} - \left\{2mg_3 F_q'^{(0)} + mg_1 F_q'^{(2)}\right\} \\
& - \frac{1}{\gamma-1}\left\{\left(\frac{2\ell+(1-m)(2q+1)+2j}{2}\right) g_5' F_q^{(0)} + \left(\frac{4m+2\ell+(1-m)(2q+1)+2j}{2}\right) F_q^{(2)} g_3' \right. \\
& \left. + \left(\frac{8m+2\ell+(1-m)(2q+1)+2j}{2}\right) F_q^{(4)} g_1'\right\} + \frac{1}{2}\left\{\left(\frac{2\ell+(1-m)(2q+1)+2j}{2}\right) F_q^{(0)} g_3'\right. \\
& + \left(\frac{4m+2\ell+(1-m)(2q+1)+2j}{2}\right) F_q^{(2)} g_1' + g_3\left(\frac{1}{\ell+1}\delta_{2q}+\delta_{1q}\right) + \frac{\gamma}{2(\gamma-1)}(f_1 g_3' + g_1' f_3)\left((\ell+m)\delta_{0q}+\delta_{1q}\right) \\
& + (g_1 f_3' + g_3 f_1')\left((m+\ell)\delta_{0q}+\delta_{1q}\right) + m(\gamma-1))(f_1' G_q^{(0)} + f_1' g_1) + G_{g-1}^{(2)} - \left(\frac{m+1+2j}{2}\right)(f_1'' F_q''^{(2)} + f_3'' F_q''^{(0)})
\end{aligned} \tag{A2}
$$

where $G_{-1}=0$.

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